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On the stability of the solution of Abel's integral equation

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Abstract. It is shown that numerical differentiation can be removed from the conventional solution of Abel's integral equation. Using the Gaussian quasipotential as an example, we show that the new inversion method is less sensitive to random errors in the input data.

1. Introduction

Abel's integral equation is the standard tool for the inversion of experimental data in different areas of physics. In atomic and molecular scattering (Buck 1974, Klingbell 1972), the direct problem is to find the phase-shift from a given potential using

$$\eta_l = k\left(\int_{r_t}^{\infty} \mathrm{d}r\,[1 - V(r)/E - (l+\tfrac{1}{2})^2/k^2r^2]^{1/2} - \int_{(l+\frac{1}{2})/k}^{\infty} \mathrm{d}r\,[1-(l+\tfrac{1}{2})^2/k^2r^2]^{1/2}\right) \tag{1}$$

where $k^2 = 2\mu E/\hbar^2$ and r_t is the classical turning point. Using the Sabatier (1965) transformation

$$s^2 = r^2(1 - V(r)/E) \tag{2}$$

and $b = (l+\frac{1}{2})/k$, one obtains Abel's integral equation

$$\eta(b) = -(\mu/\hbar^2 k)\int_b^{\infty} sQ(s)\,\mathrm{d}s/(s^2-b^2)^{1/2} \tag{3}$$

where $Q(s) = (\hbar^2k^2/\mu)\ln(r(s)/s)$. Given a set of phase-shifts, the inverse problem is to solve for the quasipotential (Vollmer 1969) $Q(s)$ from which the potential $V(r)$ can be determined. The solution of equation (3) is commonly written as

$$Q(s) = -(2/\pi s)\frac{\mathrm{d}}{\mathrm{d}s}\int_s^{\infty} \Delta(b)b\,\mathrm{d}b/(b^2-s^2)^{1/2} \tag{4}$$

or

$$Q(s) = -(2/\pi)\int_s^{\infty} \frac{\mathrm{d}\Delta(b)}{\mathrm{d}b}\,\mathrm{d}b/(b^2-s^2)^{1/2} \tag{5}$$

where $\Delta(b) = -(\hbar^2k/\mu)\eta(b)$.

In plasma spectroscopy (Fleurier and Chapelle 1974), Abel's integral equation is used to study the extended source of radiation with cylindrical symmetry. If $Y(y)$ is the transverse distribution of the intensity emitted perpendicularly to the axis of the source and $F(r)$ is the emission coefficient, then the direct problem is given by

$$Y(y)=2\int_y^R rF(r)\,\mathrm{d}r/(r^2-y^2)^{1/2} \tag{6}$$

where $Y(R)=0$ is the boundary condition. The inverse problem is solved by

$$F(r)=-(1/\pi)\int_r^R \frac{\mathrm{d}Y(y)}{\mathrm{d}y}\,\mathrm{d}y/(y^2-r^2)^{1/2}. \tag{7}$$

As shown by Di Salvo and Viano (1976), the problem of solving Abel's integral equation is improperly posed because small perturbations of the input data $\mathrm{d}\Delta(b)/\mathrm{d}b$ in equation (5) or $\mathrm{d}Y(y)/\mathrm{d}y$ in equation (7) can produce large oscillations in the solution. In practice, the noises in the original data $\Delta(b)$ or $Y(y)$ are first amplified by the numerical differentiation routine. As a result, the numerical derivatives may be too contaminated for the construction of the potential or the emission coefficient. The purpose of this paper is to remove numerical differentiation from the conventional solution of Abel's integral equation. As a result, we obtain the solution in another form which is numerically more stable.

2. Solution of Abel's integral equation without numerical differentiation

Let us start from equation (4) with the substitutions $s^2=q$ and $b^2=p$; then we have $U(q)=Q(s)$, $\Delta(b)=\delta(p)$ and therefore

$$U(q)=(-2/\pi)\frac{\mathrm{d}}{\mathrm{d}q}\int_q^\infty \delta(p)\,\mathrm{d}p/(p-q)^{1/2}. \tag{8}$$

To remove the apparent singularity at $p=q$, we let $u^2=p-q$; then

$$U(q)=(-4/\pi)\int_0^\infty \frac{\partial}{\partial q}(\delta(q+u^2))\,\mathrm{d}u. \tag{9}$$

Since $(\partial/\partial q)\delta(q+u^2)=(1/2u)(\partial/\partial u)\delta(q+u^2)$, we have

$$U(q)=(-2/\pi)\int_0^\infty (1/u)\,\mathrm{d}(\delta(q+u^2)). \tag{10}$$

Integrating by parts, we obtain

$$U(q)=(-2/\pi)\lim_{\epsilon\to 0}\left(-\delta(q)/\epsilon+\int_\epsilon^\infty \delta(q+u^2)\,\mathrm{d}u/u^2\right). \tag{11}$$

If $\delta(q)/\epsilon$ is rewritten as an integral, we have the final result

$$U(q)=(-2/\pi)\int_0^\infty (\delta(q+u^2)-\delta(q))\,\mathrm{d}u/u^2. \tag{12}$$

The only flaw in equation (12) is the removable singularity at $u=0$. Nevertheless, we can handle this integrable singularity by extrapolation (Davis and Rabinowitz 1967).

3. An example: construction of the Gaussian quasipotential

As a simple example, we consider the Gaussian quasipotential $Q(s)=\exp(-s^2)$ and the corresponding $\Delta(b)$ is given by $\frac{1}{2}\pi^{1/2}\exp(-b^2)$. Using $\delta(p)=\frac{1}{2}\pi^{1/2}\exp(-p)$ in equation (12) and the formula

$$\int_\epsilon^\infty \exp(-x^2)\,dx/x^2=\exp(-\epsilon^2)/\epsilon-\pi^{1/2}\left(1-(2/\pi^{1/2})\int_0^\epsilon \exp(-x^2)\,dx\right), \qquad (13)$$

we can recover the quasipotential $U(q)=\exp(-q)$ analytically.

To study the numerical stability of equations (9) and (12), we calculated $U(q)=\exp(-q)$ from 26 data points

$$\delta(p_i)=\tfrac{1}{2}\pi^{1/2}\exp(-p_i) \qquad (14)$$

where $p_i=0{\cdot}2(i-1)$ and i runs from 1 to 26. These input data were first contaminated by random errors of the orders 10^{-2} and 10^{-3} respectively. Then they were differentiated and integrated accordingly using natural cubic splines assuming unknown end conditions. Numerical results in tables 1 and 2 show that equation (12) is more reliable.

Table 1. Construction of the quasipotential $U(q)$ from input data with errors of the order 10^{-2}.

q	Exact $U(q)$	$U(q)$ from equation (9)	$U(q)$ from equation (12)
0·0	1·000 0	0·702 5	1·034 4
0·6	0·548 8	0·615 8	0·567 7
1·0	0·367 9	0·417 1	0·380 5
1·6	0·201 9	0·190 7	0·208 8
2·0	0·135 3	0·068 4	0·139 9
3·0	0·049 8	0·035 6	0·051 5
4·0	0·018 3	0·019 8	0·018 9
5·0	0·006 74	0·006 67	0·006 97

Table 2. Construction of the quasipotential $U(q)$ from input data with errors of the order 10^{-3}.

q	Exact $U(q)$	$U(q)$ from equation (9)	$U(q)$ from equation (12)
0·0	1·000 0	0·917 9	1·003 3
0·6	0·548 8	0·556 3	0·550 6
1·0	0·367 9	0·372 7	0·369 1
1·6	0·201 9	0·200 7	0·202 5
2·0	0·135 3	0·128 5	0·135 8
3·0	0·049 8	0·048 3	0·049 9
4·0	0·018 32	0·018 46	0·018 37
5·0	0·006 74	0·007 08	0·006 76

In conclusion, in the absence of numerical differentiation, the solution of Abel's integral equation is less susceptible to errors in the input data.

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